

DOCUMENT RESUME

ED 047 998

24n

SE 010 881

AUTHOR Fehr, Howard F.
 TITLE Secondary School Mathematics Curriculum Improvement Study. Final Report.
 INSTITUTION Secondary School Mathematics Curriculum Improvement Study, New York, N.Y.
 SPONS AGENCY Office of Education (DHEW), Washington, D.C. Bureau of Research.
 BUREAU NO ER-7-0711
 PUB DATE Aug 70
 CONTRACT OEC-1-7-070711-4420
 NOTE 23p.

EDRS PRICE MF-\$0.65 PC-\$3.29
 DESCRIPTORS *Curriculum Development, *Instruction, Mathematics Education, *Projects, *Secondary School Mathematics

ABSTRACT

This report describes the planning, writing, evaluation, and results and recommendations of the Secondary School Mathematics Curriculum Improvement Study. This study was begun "to formulate and test a unified secondary school mathematics program (7-12) that will take capable students well into current collegiate mathematics" and to "determine the education required by teachers who will implement such a program." Also included in the report is a scheme for a taxonomy of objectives and course contents for each of the three courses developed (grades 7-8-9). Recommendations are that the program be reexamined for possible adoption as a curriculum for all junior high school students, that the preservice mathematics education of prospective junior high school teachers be reexamined, and that the unified approach developed for junior high school be extended throughout the senior high school. (FL)

ED047998

BR 7-0
PA 24
SE

FINAL REPORT
Project No. 7-0711
Contract No. OEC 1-7-070711-4420

SECONDARY SCHOOL MATHEMATICS CURRICULUM
IMPROVEMENT STUDY

August, 1970

U.S. DEPARTMENT OF HEALTH, EDUCATION
& WELFARE
OFFICE OF EDUCATION
THIS DOCUMENT HAS BEEN REPRODUCED
EXACTLY AS RECEIVED FROM THE PERSON OR
ORGANIZATION ORIGINATING IT. POINTS OF
VIEW OR OPINIONS STATED DO NOT NECESS-
ARILY REPRESENT OFFICIAL OFFICE OF EDU-
CATION POSITION OR POLICY.

U.S. DEPARTMENT OF
HEALTH, EDUCATION, AND WELFARE

Office of Education
Bureau of Research

FINAL REPORT
Project No. 7-0711
Contract No. OEC 1-7-070711-4420

SECONDARY SCHOOL MATHEMATICS CURRICULUM
IMPROVEMENT STUDY

Howard F. Fehr
Teachers College, Columbia University
New York, New York

August, 1970

The research reported herein was performed pursuant to a contract with the Office of Education, U.S. Department of Health, Education, and Welfare. Contractors undertaking such projects under Government sponsorship are encouraged to express freely their professional judgment in the conduct of the project. Points of view or opinions stated do not, therefore, necessarily represent official Office of Education position or policy.

U.S. DEPARTMENT OF
HEALTH, EDUCATION, AND WELFARE

Office of Education
Bureau of Research

CONTENTS

	PAGE
Acknowledgments	111
Summary	1
Introduction	2
Planning the 7-12 Program	3
Writing of Courses I, II, and III	3
Education of Teachers	4
Teaching Courses I, II, and III	6
Evaluation of Courses I, II, and III	6
Future Activity	7
Conclusions and Recommendations	8
Table I (Scheme for Taxonomy of Objectives)	10
Appendix A (Course I Contents)	11
Appendix B (Course II Contents)	15
Appendix C (Course III Contents)	18

ACKNOWLEDGMENTS

Planning, writing, and teaching of SSMCIS Courses I, II, and III was done with the cooperation of the schools and teachers in the Metropolitan New York Area and the following consultants:

Nicholas A. Branca, Teachers College, Columbia University
John Camp, Teachers College, Columbia University
Gustave Choquet, Universite de Paris, France
Ray Cleveland, University of Calgary, Canada
John Downes, Emory University
Howard F. Fehr, Teachers College, Columbia University
James Fey, University of Maryland
Alan Gewirtz, City University of New York
Abraham Glicksman, Bronx High School of Science, New York
Richard Good, University of Maryland
Vincent Haag, Franklin and Marshall College
Thomas Hill, University of Oklahoma
Peter Hilton, Cornell University
Julius Hlavaty, National Council of Teachers of Mathematics
Michael Hoban CFC, Iona College, New York
Meyer Jordan, City University of New York
Burt Kaufman, Southern Illinois University
Howard Kellogg, Teachers College, Columbia University
Jeremy Kilpatrick, Teachers College, Columbia University
Erik Kristensen, Aarhus University, Denmark
Howard Levi, City University of New York
Edgar R. Lorch, Columbia University
Richard C. Pocock, Houghton College, New York
Lennart Rade, Chalmers Institute of Technology, Sweden
Myron F. Roskopf, Teachers College, Columbia University
Harry Ruderman, Hunter College High School, New York
Harry Sitomer, C.W. Post College
Hans-Georg Steiner, University of Karlsruhe, Germany
Marshall H. Stone, University of Massachusetts
Stanley Taback, New York University
H. Laverne Thomas, State University College at Oneonta, N.Y.
Albert W. Tucker, Princeton University
Bruce Vogeli, Teachers College, Columbia University
Lucian Wernick, Illinois Western State College

SUMMARY

The Secondary School Mathematics Curriculum Improvement Study (SSMCIS) has two main objectives:

- 1) To formulate and test a unified secondary school mathematics program (7 - 12) that will take capable students well into current collegiate mathematics;
- 2) To determine the education required by teachers who will implement such a program.

To inaugurate the study, leading United States and European mathematicians and educators met in June 1966 to formulate a position paper stating the aims and procedures of the study, to construct a flow charted analysis of the proposed 7 - 12 mathematics courses, and to make detailed recommendations for the mathematical content of Course I. Using this detailed syllabus as a guide, a team of eight mathematics educators wrote a textbook for Course I during that summer. Each chapter was written by one writer, reviewed by the other writers and a consulting mathematician, and then revised for printing. Teachers' guides and solutions to exercises were written and distributed to the teachers.

In each subsequent year (1967, 1968) a two-week June working conference was held to review and revise the previous year's experimental text and to make specific recommendations for the content and teaching of a new course—Course II and Course III, respectively. As in the first year the writing team used the chapter guides developed by the June working conference to write the new texts for the following year.

Six junior high schools in the metropolitan New York area have participated in the experimental teaching of Courses I, II, and III from the initiation of the experiment. Each of these schools designated a team of two capable and interested teachers who taught all the pilot classes using the experimental textbooks. Each summer, while new materials were being written, six weeks of instruction was given to these teachers in preparation for teaching the new SSMCIS course. This instruction included 50 hours in the fundamental concepts underlying the unified mathematics program and 50 hours in contemporary methods of teaching those concepts.

The experimental teaching was evaluated in three ways. The director and project staff members made frequent visits to the classes for direct observation. The students were tested by examinations - prepared by the project staff - designed specifically to measure learning of important new concepts introduced in the courses. Teachers, staff, and consultants met at full day conferences to discuss progress and problems in the experimental teaching.

Results of the experimental teaching have shown that the new mathematics courses, based on fundamental concepts and structures, give promise of meeting the expectations of the proposed six year program.

Introduction

During the past decade the United States has been engaged in revising the elementary and secondary school mathematics curriculum - primarily by updating the existing traditional curriculum. Modest recommendations of the Commission on Mathematics have been largely accepted by curriculum and syllabus bodies and by writers of commercially produced textbooks. Implementation of this program by the SMSG has had wide acceptance and massive experimental use throughout the country.

Throughout all of our reform movements the traditional division of mathematics instruction into separate years of arithmetic, algebra, and geometry has been maintained. Beyond introduction of new concepts, little has been gained in bringing more advanced study into the high school through more efficient methods of organizing the subject matter. Bolder and more radical recommendations for the improvement of secondary school education in mathematics have been made both in this country, notably by the UICSM, and in Europe, notably in Belgium, Switzerland, and Denmark.

What has been called for is reconstruction of the entire curriculum from a global point of view - one which eliminates the barriers separating the several branches of mathematics and unifies the subject through its general concepts (sets, operations, mappings, and relations) and builds the fundamental structures of the number systems, algebra, and geometry (groups, rings, fields, and vector spaces). The efficiency gained by such organization should permit introduction into the high school program of much that was previously considered undergraduate mathematics.

In September 1965, the Commissioner of Education, Department of Health, Education, and Welfare, Office of Education, approved for support for a period of 18 months the Secondary School Mathematics Curriculum Improvement Study (SSMCIS), an experimental study whose objective would be the construction of a unified school mathematics curriculum for grades seven through twelve.

In June, 1967, continuation of this support was granted for an additional 36-month period ending June 30, 1970. This is a report of the activities and findings of the SSMCIS during this latter period, covering the writing and teaching of experimental Courses II and III, the revision and further, teaching of Courses I and II, and the final revision of Courses I and II.

Planning the 7 - 12 Program

In June 1966 a group of eighteen leading United States and European mathematicians and educators met for 20 days to outline the scope and sequence of a six year unified secondary school mathematics program. The first half of the conference was devoted to producing a complete flow charted analysis of the proposed course. Then topics planned for the seventh grade were expanded in working papers which outlined the mathematical content of each textbook chapter and made specific suggestions for writing and teaching these ideas.

Writing of Courses I, II, and III

During July and August 1966, a team of eight mathematical educators wrote the textbook for Course I, using the syllabus produced in June as a guide. Each textbook chapter was written by one writer, reproduced for review by the other writers and consulting mathematicians, and then rewritten, incorporating the reviewers' suggestions. Teachers' guides and solutions to exercises were written for each chapter. These notes, mimeographed and distributed to teachers of experimental classes, included discussions of fundamental mathematical ideas underlying each chapter, hints for possible class activity to accompany reading of the text, and suggested time allotment to the various topics. The Course I textbook was then published in three volumes.

To initiate the detailed planning for Course II and for the revisions of Course I, a pre-planning session was held on March 11-12, 1967. The recommendations of this pre-planning group were considered in detail at a working conference held in June, 1967, at which the full group of

writers and consultants was present. This group produced a detailed set of writing guidelines, which were then used by the summer writing teams to produce the text materials for experimental Course II and to revise Course I.

Beginning with a pre-planning meeting in December, 1967, and followed by a full working conference in June, 1968, a similar procedure was used to plan and write experimental Course III, to rewrite Course II, and to finalize Course I. In addition, in order to make the fullest possible use of the experience of two years of teaching Course I, ten of the experimental teachers re-wrote and expanded the original teachers' commentaries for Course I, which had been originally written by the authors of the various chapters. These chapter-by-chapter commentaries were then bound into a single volume and made available for public use along with the texts.

In 1969, the writing activities of the SSMCTS, supported by the Office of Education, consisted of rewriting Course III and making final revisions in Course II. Again, as in the previous year, a group of teachers of experimental classes worked during the summer, to rewrite the teachers' commentaries for Course II, which were then printed in a single volume for use in conjunction with the texts.

The texts and Teachers Commentaries for Courses I and II (1 each in two volumes) are now available for widespread classroom use, and are published and distributed by the Teachers College Press. When revision of Course III, now under way, is completed, the two volumes of the Course III text and the accompanying Teachers Commentary will be similarly available.

Education of Teachers

Each summer, beginning in 1966, the teachers of experimental classes have participated in a six-week program of special study at Teachers College, designed to prepare them to teach an experimental course in the following school year. This program of study was in two parts. The first was a course in the mathematical subject areas underlying the experimental course content, such as: abstract algebra, linear algebra, transformation geometry and probability. These subjects were taught from a modern and unified point of view. The second part of this program was a course in pedagogical methods of teaching secondary mathematics as a unified branch of knowledge, with special emphasis on the specific structures and principles to be covered in the following year of experimental teaching.

The following is a list of the teachers and the schools in which they taught the experimental classes (Course III) during the 1968-69 school year:

Elmont, New York

Schools: Alva T. Stanforth Junior High School
Sewanhaka High School

Teachers: Samuel Backer
Alexander Imre
Edward Keenan
Mary Murray

Leonia, New Jersey

School: Leonia High School
Teachers: Christine McGoey
David Swaim

New York, New York

School: Hunter College High School
Teachers: Douglas Bumby
Ruth Cohen
Richard Klutch

Teaneck, New Jersey

Schools: Benjamin Franklin Junior High School
Thomas Jefferson Junior High School
Teaneck High School
Teachers: Franklin Armour
Annabelle Cohen
Otto Krupp
Mary Renda

Westport, Connecticut

Schools: Bedford Junior High School
Coleytown Junior High School
Long Lots Junior High School
Staples High School
Teachers: David Fuys
Robert Keller
John Pepe
Daniel Sullivan

All teachers showed intense interest and cooperated splendidly in acquiring the spirit and content of the proposed new curriculum, and in teaching it. As a result of this training we now have a core of demonstration teachers and also a body of subject matter that must constitute teacher preparation in the future.

Teaching Courses I, II, and III

Six junior high schools in the New York Metropolitan area have taken experimental classes through Course I, II, and III. Five of these classes covered the text material in its first experimental revision. Another fifteen experimental classes have completed these three courses in the revised versions and another fifteen non-experimental classes in these schools have completed both Course I and II in their final revisions. Since the SSMCIS program is at present designed for those students in roughly the top 15% of their class with respect to mathematical ability, the original selection of students for the twenty experimental classes was made by the participating schools with prior mathematics achievement and scores on aptitude tests as main criteria.

Because the teachers of pilot classes were working as a team in the experimental class, they were often able to help each other with difficulties that arose in understanding or teaching the new material. Having had this year of team teaching experience, the teachers are now prepared to teach Courses I, II, and III on their own.

During the school year, the director and project staff members made frequent personal visits to observe the experimental teaching. Each class was observed at least four times. Visits to these schools included discussions with the teachers and administrators concerning progress and problems with the experimental course.

The teachers were further assisted by several full-day meetings at Teachers College where teaching problems were reviewed with selected consultants and the project director. At these meetings many teaching difficulties were resolved and valuable criticisms of the textbook were gathered.

Evaluation of Courses I, II, and III

The six year mathematics program introduces many new concepts into the secondary school mathematics curriculum and integrates both standard and new topics in a global organization not characteristic of existing programs. Student achievement in such a program cannot adequately be measured using conventional standardized tests. For this reason, student learning was tested by extramural examinations constructed by the project staff.

To guide construction of these and future measurement instruments, the textbooks were analyzed to produce a taxonomy of cognitive objectives. This taxonomy aided in

delineating goals of instruction in terms of subject matter and related behaviors. The categories of behavior appear in Table I.

Each year, two examinations were prepared; one for a mid-year evaluation, and one for an evaluation at the end of the year. These instruments were used as a measure of the teachability as well as of the learning of the prepared content. They also guided the revisions that were subsequently made.

Although achievement on standardized traditional mathematics tests was not accepted as a measure of the success of the experimental program, it was of interest to determine at the start whether or not study in the experimental Course I affected learning of traditional topics. To accomplish this objective all students were administered the Sequential Test of Educational Progress - Mathematics, Form 3A - in September 1966 and again in September 1967. The test results clearly show that students in the project classes suffered no decline in mathematical skills when compared with students studying more traditional programs. Moreover, the achievement of these students on the project tests shows that they were learning to work with many new and powerful mathematical tools not a part of the traditional mathematics fare of seventh graders.

Future Activity

The Secondary School Mathematics Curriculum Improvement Study received support from the National Science Foundation in June 1969 to continue designing and experimenting with a unified secondary school program for college capable students in the senior high school. This support has enabled the project to produce Course IV in the six year sequence and to begin planning for Course V and VI.

Course IV, which had been written in the summer of 1969, was pilot-tested in five experimental schools during the 1969-1970 school year. As a result of this experimentation, Course IV will be revised during the summer of 1970 and will be available by the early Fall (1970). After further testing, the final version will be available by September 1971.

A tentative outline for Course V was developed during a two day conference of the advisory council in January 1970. This outline was expanded during a nine day conference of writers and consultants into detailed guides for the writers of the chapters in Course V. This course will be written during the summer of 1970 and experimented with in five schools during 1970.

The planning for and writing of Course VI will follow a similar procedure to that outlined for Course V with classroom testing during the 1971-72 school year.

By the end of the academic year 1973, the SSMCIS will have completed its task - to make a reconstruction of the secondary mathematics curriculum by presenting the subject as an integrated body of knowledge reflecting the spirit of contemporary mathematics.

Conclusions and Recommendations

After 5 years of classroom experimentations with students selected in the upper 15 to 20% of academic ability and taught by interested classroom teachers with special training in subject matter and pedagogy, a new curriculum in mathematics has been designed for the junior high school study, grades seven, eight, and nine. This curriculum breaks down the traditional barriers separating arithmetic, algebra, and geometry, and unifies the study through those fundamental concepts underlying all the branches, namely sets, relations, functions, and operations. The resulting curriculum is like a double helix in which the important structures - group, ring, field and vector space - form one strand, while the other strand consists of the important realizations: the number systems and the several geometries; synthetic, coordinate, vector, and transformation. Interwoven with both these strands are the activities and applications including the study of function, conditional sentences, statistics and probability.

Students who complete this three year program are advanced in knowledge more than one year beyond the present college preparatory program. This is accomplished by eliminating a great deal of traditional content that today is of little or no value in further study or application of mathematics. The increase in learning is also brought about by the unification of all the study under the more general concepts and structures of contemporary mathematics.

The teachability has been tested and verified, each course undergoing three years of thorough examinations and revision into its present form. Concomitantly, five doctoral studies researching the learning and teaching aspects of the program were completed. These studies were carried out by the research assistants associated with the project. They were:

Nicholas A. Branca - "Strategies in Learning Mathematical Structures," 1970.

James T. Fey - "Patterns of Verbal Communication in Mathematics Classes," 1968.

Michael J. Hoban - "Transformation Geometry in the Junior High School: An Evaluation of a Curricular Unit in the 7th Grade," 1970.

Stanley F. Taback - "The Child's Concept of Limit," 1969.

H. Laverne Thomas - "An Analysis of Stages in the Attainment of a Concept of Function," 1969.

A study in one school has shown that the same material can be learned by students of average ability, if pursued at a slower rate over a longer period of time.

It is strongly recommended that the program developed by SSMCIS be reexamined for adaptation as a curriculum for all students in the junior high school. It is also recommended that the SSMCIS program serve as a basis for re-examining the pre-service mathematics education of prospective junior high school teachers of mathematics. For these teachers, far more attention must be given to abstract and linear algebra, and to geometry of a modern variety, with less stress on advanced analysis. Lastly it is recommended that the unified approach be extended throughout the senior high school study to give college preparatory students an advanced knowledge of all aspects of mathematics on entering college and not only that of the calculus.

TABLE I
TAXONOMY OF OBJECTIVES

Mathematical Objectives

Structures:	Arithmetic	Geometry	Probability	Analysis
	and Algebra			
Fundamental Concepts	Sets	Operations	Relations	Mappings
				Logic

Behavioral Objectives

- I. Ability to recall definitions, notations, operations, concepts.
- II. Ability to manipulate and calculate efficiently.
- III. Ability to interpret symbolic data or processes.
- IV. Ability to communicate mathematical ideas.
- V. Ability to apply concept to a purely mathematical situation--solve problems.
- VI. Ability to apply concept to problems in other situations--solve word problems.
- VII. Ability to transfer learning to a new situation in mathematics.
- VIII. Ability to construct or follow a mathematical argument.

Of course not all these categories apply to each subject matter topic, but the goals were checked against subject matter.

APPENDIX A
COURSE I CONTENT

Chapter

- 1 FINITE NUMBER SYSTEMS
 Jane Anderson's Arithmetic
 Clock Arithmetic
 $(\mathbb{Z}, +)$ and $(\mathbb{W}, +)$
 Calendar Arithmetic
 Open Sentences
 New Clocks
 Rotations
 Subtraction in Clock Arithmetic
 Multiplication in Clock Arithmetic
 Comparison of (\mathbb{W}, \cdot) and Clock Multiplication
 Division in Clock Arithmetic
 Inverses in Clock Arithmetic
 The Associative and Distributive Properties
 Summary
- 2 SETS AND OPERATIONS
 Ordered Pairs of Numbers and Assignments
 What is an Operation?
 Computations with Operations
 Open Sentences
 Properties of Operations
 Cancellation Laws
 Two Operational Systems
 What is a Group?
 Summary
- 3 MATHEMATICAL MAPPINGS
 Assignments and Mappings
 Mappings of Sets of Whole Numbers
 Mappings of Clock Numbers
 Sequences
 Composition of Mappings
 Inverse and Identity Mapping
 Special Mappings of \mathbb{W} to \mathbb{W}
 Summary
- 4 THE INTEGERS AND ADDITION
 Introduction
 Some New Numbers
 The Integers and Opposites
 Properties of $(\mathbb{Z}, +)$
 The Integers and Translations on a Line
 Subtraction in $(\mathbb{Z}, +)$
 Subtraction as Addition of Opposites
 Equations in $(\mathbb{Z}, +)$
 Cancellation Law

Chapter

Ordering the Integers
Absolute Value
Summary

- 5 PROBABILITY AND STATISTICS
 Introduction
 Discussion of an Experiment
 Experiments to be Performed by Students
 The Probability of an Event
 A Game of Chance
 Equally Probable Outcomes
 Another Kind of Mapping
 Counting with Trees
 Preview
 Research Problems
 Statistical Data
 Presenting Data in Tables
 The Frequency Histogram and the Cumulative
 Frequency Histogram
 Summary

- 6 MULTIPLICATION OF INTEGERS
 Operational Systems $((W, \cdot)$ and (Z, \cdot)
 Multiplication for Z
 Multiplication of a Positive Integer and a
 Negative Integer
 The Product of Two Negative Integers
 Multiplication of Integers through Distributivity
 Dilations and Multiplication of Integers
 Summary

- 7 LATTICE POINTS IN THE PLANE
 Lattice Points and Ordered Pairs
 Conditions on $Z \times Z$ and their Graphs
 Intersection and Unions of Solution Sets
 Absolute Value Conditions
 Lattice Point Games
 Sets of Lattice Points and Mappings of Z into Z
 Lattice Points in Space
 Translation and $Z \times Z$
 Dilations and $Z \times Z$
 Some Additional Mappings and $Z \times Z$
 Summary

- 8 SETS AND RELATIONS
 Sets
 Set Equality and Subsets
 Universal Set, Subsets and Venn Diagrams
 Unions, Intersections and Complements
 Cartesian Product Sets: Relations
 Properties of Relations
 Equivalence Classes and Partitions
 Summary

Chapter

9

TRANSFORMATIONS OF THE PLANE
Knowing How and Doing
Reflections in a Line
Lines, Rays and Segments
Perpendicular Lines
Rays Having the Same Endpoint
Reflection in a Point
Translations
Rotations
Summary

10

SEGMENTS, ANGLES, AND ISOMETRIES
Introduction
Lines, Rays, Segments
Planes and Halfplanes
Measurements of Segments
Midpoints and other Points of Division
Using Coordinates to Extend Isometries
Coordinates and Translations
Perpendicular Lines
Using Coordinates for Line and Point Reflections
What is an Angle?
Measuring an Angle
Boxing the Compass
More about Angles
Angles and Line Reflections
Angles and Point Reflections
Angles and Translations
Sum of Measures of the Angles of a Triangle
Summary

11

ELEMENTARY NUMBER THEORY
($N, +$) and (N, \cdot)
Divisibility
Primes and Composites
Complete Factorization
The Sieve of Eratosthenes
On the Number of Primes
Euclid's Algorithm
Summary

12

THE RATIONAL NUMBERS
 W , Z and Z ,
Reciprocals of the Integers
Extending $Z \cup Z'$ to Q
(Q, \cdot)
Properties of (Q, \cdot)
Division of Rational Numbers
Addition of Rational Numbers

Chapter

Subtraction of Rational Numbers
Ordering the Rational Numbers
Decimal Fractions
Infinite Repeating Decimals
Decimal Fractions and Order of the Rational Numbers
Summary

- 13 SOME APPLICATIONS OF THE RATIONAL NUMBERS
Rational Numbers and Dilations
Computation with Decimal Fractions
Ratio and Proportion
Using Proportions
Meaning of Percent
Solving Problems with Percents
Presenting Data in Rectangular, Circle, and
 Bar Graphs
Translations and Groups
Applications of Translations
Summary

- 14 ALGORITHMS AND THEIR GRAPHS
Planning a Mathematical Process
Flow Charts of Branching Algorithms
Interactive Algorithms
Truncated Routines and Truncation Criteria
Summary

APPENDIX B
COURSE II CONTENT

Chapter

- 1 MATHEMATICAL LANGUAGE AND PROOF
 Introduction
 Mathematical Statements
 Connectives: And, Or
 Conditional and Bi-conditional Statements
 Quantified Statements
 Substitution Principle for Equality (SPE)
 Inference
 Direct Mathematical Proof
 Indirect Mathematical Proof
 Summary
- 2 GROUPS
 Definition of a Group
 A Non-Commutative Group
 More on Permutations
 Functional Notation
 More Notation
 Some Theorems About Groups
 Isomorphism
 Summary
- 3 AN INTRODUCTION TO AXIOMATIC AFFINE GEOMETRY
 Preliminary Remarks
 Axioms
 Some Logical Consequences of the Axioms
 A Non-Geometric Model of the Axioms
 Other Models of the Axioms - Finite and Infinite
 Equivalence Classes of Parallel Lines
 Parallel Projection
 Vectors - An Intuitive Introduction
 Summary
- 4 FIELDS
 What is a Field?
 Getting Some Field Theorems Painlessly
 Trouble with 0
 Subtraction and Division in Fields
 Fractions in Fields
 Order in Fields
 How Many Ordered Fields?
 Equations and Inequations in $(Q, +, \cdot, <)$
 Solving Quadratic Equations
 Summary

Chapter

- 5 THE REAL NUMBER SYSTEM
The Equation $x^2 = 2$ in $(Q, +, \cdot)$
The Measuring Process
The Length of a Line Segment
Three Illustrative Cases
The Real Number System
Some Properties of the Real Number System
Arithmetic of Irrational Numbers
Summary
- 6 COORDINATE GEOMETRY
Introduction
Axiom 4. Uniqueness of Line Coordinate Systems
Axiom 5. Relating Two Coordinate Systems on a Line
Segments, Rays, Midpoints
Axiom 6. Parallel Projections and Line Projections
Plane Coordinate Systems
An Equation for a Line
Intersections of Lines
Triangles and Quadrilaterals
The Pythagorean Property
Plane Rectangular Coordinate Systems
Summary
- 7 REAL FUNCTIONS
Mathematical Mappings
Properties of Real Functions
Representing Real Functions
Composition of Real Functions
Inverses of Real Functions
 $[f + g]$ and $[f - g]$
 $[f \cdot g]$ and $[\frac{f}{g}]$
The Square Root and Cube Root Functions
Summary
- 8 DESCRIPTIVE STATISTICS
Introduction
Examples of Sets of Data and Their Graphical
Presentation
The Symbol Σ and Summation
The Arithmetic Mean, Its Computation and Properties
Measures of Dispersion
Simplified Computation of the Variance and the
Standard Deviation
Summary

Chapter

- 9 TRANSFORMATIONS IN THE PLANE: ISOMETRIES
What is a Transformation?
Reflections in a Line
Translations
Rotations and Half-Turns
Composing Isometries, Glide Reflections
The Three Line Reflection Theorem
Directed Isometries
Groups of Isometries
Isometry, Congruence, and Symmetry
Other Transformations: Dilations and Similarities
Summary

- 10 LENGTH, AREA, VOLUME
Introduction
Measures on Sets
Lengths of Line Segments
Areas of Rectangular Regions
Volumes of Rectangular Solids
Areas of Triangular Regions
Areas of Parallelograms and Trapezoidal Regions
Areas of other Regions
Circumference of a Circle and π
Areas of Circular Regions
Summary

- Appendix A: Mass Points
Mass Points
Notations and Procedures
Axioms for Mass Points
A Theorem
Another Theorem
Using a Definition
Mass Points in Space and a Theorem
Summary

APPENDIX C
COURSE III CONTENT

Chapter

- 1 INTRODUCTION TO MATRICES
What is a Matrix
Using Matrices to Describe Complex Situations
Operations on Matrices
Matrices and Coded Messages
Matrices and Transformations
Transition Matrices
Summary
- 2 LINEAR EQUATIONS AND MATRICES
Linear Combinations of Equations
Solving Systems of Linear Equations
Solving Systems of Linear Equations, Continued
Homogeneous Linear Equations
Systems of Linear Equations and Matrices
Matrix Inversion
Summary
- 3 ALGEBRA OF MATRICES
The World of Matrices
Addition of Matrices
Multiplication by a Scalar
Multiplication of Matrices
Multiplicative Inverses in M_n
The Ring of 2×2 Matrices
A Field of 2×2 Matrices
Summary
- 4 GRAPHS AND FUNCTIONS
Conditions and Graphs
Regions of the Plane and Translations
Functions and Conditions
Functions and Solution of Equations
Operations on Functions and Asymptotes
Summary
- 5 COMBINATORICS
Introduction
Counting Principle and Permutations
The Power Set of a Set
Number of Subsets of a Given Size
The Binomial Theorem
Mathematical Induction
Summary

Chapter

- 6 PROBABILITY
 Introduction
 Outcome Set and Events
 Probability Measure
 Uniform Probability Measure
 Looking Back
 Looking Ahead
 Summary
- 7 POLYNOMIAL AND RATIONAL FUNCTIONS
 Polynomial Functions
 Degree of a Polynomial
 Addition of Polynomials $(P,+)$
 Multiplication of Polynomial Functions $(P,+,\cdot)$
 Division of Polynomial Functions
 Polynomial Factors and The Factor Theorem
 Quadratic Functions and Equations
 Rational Functions
 Operations with Real Rational Functions
 Summary
- 8 CIRCULAR FUNCTIONS
 Sensed Angles
 Standard Position
 Circular Functions of Angles
 Circular Functions of Real Numbers
 Degree Measure and Special Angles
 Graphs of Circular Functions
 Law of Cosines and Law of Sines
 Summary
- 9 INFORMAL SPACE GEOMETRY
 Space Geometry and Plane Geometry
 Planes in Space
 Parallel Lines and Planes in Space
 Deductive Approach to Geometry in Space
 Perpendicularity of Lines and Planes in Space
 Coordinate Systems in 3-Space
 Set Descriptions of Planes in Coordinate 3-Space
 Surfaces in Space
 Summary